JOURNAL OF APPROXIMATION THEORY 13, 40–44 (1975)

Lebesgue Constants for Spherical Partial Sums

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Communicated by P. L. Butzer

DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

We use conventional vector notations of multidimensional Fourier analysis, as in [5]. Let $f \in L^1(T^m)$, $T = [0, 2\pi)$ (always tacitly assumed to be extended to all of R^m, 2π-periodic in each variable). For $n = (n_1, ..., n_m) \in Z^m$, the Fourier coefficient of rank n of f is

$$\hat{f}(n) = (2\pi)^{-m} \int_{T^m} f(t) e^{-in\cdot t}\, dt.$$

The spherical partial sum of rank $N > 0$ of the Fourier series of f is

$$S_N(f, t) = \sum_{|n| \leqslant N} \hat{f}(n) e^{in\cdot t}, \qquad |n| = \left(\sum_{j=1}^{m} n_j^2\right)^{1/2}.$$

This is the convolution $f * D_N$ of f with the corresponding *spherical Dirichlet kernel*

$$\Delta_N^{(m)}(t) = \sum_{|n| \leqslant N} e^{in\cdot t}$$

and various convergence questions, especially that of the uniform convergence of S_N for $f \in C(T^m)$, depend upon estimates for the *spherical Lebesgue constants*:

$$\Lambda_N^{(m)} = \| \Delta_N^{(m)} \|_{L^1(T^m)}.$$

So far as I am aware, the only studies devoted to $\Lambda_N^{(m)}$ for $m \geqslant 2$ to have appeared in print are those of Mitchell [2, 3]. She proved in [2], using intricate arguments based upon refined estimates for the distribution of lattice points, that $\Lambda_N^{(2)} = O(N^{2/3})$ and $\Lambda_N^{(3)} = O(N^{4/3}(\log N)^{5/3})$. In [3] she asserted that a more precise analysis yields $\Lambda_N^{(2)} = O(N^{1/2})$, but did not give details. As remarked in [2], we also have the estimate $\Lambda_N^{(m)} = O(N^{m/2})$ for all m, by a simple use of the Schwartz inequality. The purpose of this note is to prove, by a new and simple method,

THEOREM. $\Lambda_N^{(m)} = O(N^{(m-1)/2})$ *for* $m \geqslant 2$.

Remark. The exponent $(\frac{1}{2})(m-1)$ is sharp. This is a consequence of results of V. P. Il'in in Chapter 3 of [1] (see especially p. 93 of the English version). Il'in does not restrict himself to Fourier series, but studies more general eigenfunction expansions. There is little doubt that the above theorem can be deduced also from Il'in's results, however the following method of proof seems to be of interest, especially since it applies, with minor modifications, to summation methods based on convex bodies other than spheres.

The essence of the theorem is contained in the following lemma, whereby $A = A(\hat{R}^m)$ denotes as usual the set of Fourier transforms of functions in $L^1(R^m)$, with $\|\hat{f}\|_A$ defined to be the norm of f in $L^1(R^m)$.

LEMMA. *Let* $m \geqslant 2$ *and* $N \geqslant 1$. *There is a function* $F = F_N \in A(\hat{R}^m)$ *such that*

(i) $F(x) = 1, \; |x| \leqslant N,$

(ii) $F(x) = 0, \; |x| \geqslant N+1,$

(iii) $0 \leqslant F(x) \leqslant 1,$

(iv) $\|F\|_A \leqslant CN^{(m-1)/2},$

where C *depends only on* m.

Assuming this, let us show how the theorem follows. As is well known (see, e.g., Stein and Weiss [5, p. 251] the restriction $F \mid Z^m$ is the sequence of Fourier coefficients of a function $\varphi \in L^1(T^m)$ whose $L^1(T^m)$ norm does not exceed $\|F\|_A$. But

$$\varphi(t) = \Delta_N^{(m)}(t) + \sum_{N<|n|\leqslant N+1} \hat{\varphi}(n)e^{in\cdot t}$$

and the latter sum has $L^2(T^m)$ norm, and a fortiori $L^1(T^m)$ norm, not exceeding $C'N^{(m-1)/2}$. Hence,

$$\Lambda_N^{(m)} \leqslant \|\varphi\|_{L^1(T^m)} + C'N^{(m-1)/2} \leqslant C''N^{(m-1)/2},$$

where C', C'' depend only on m. This proves the theorem.

Proof of lemma. Let G denote the characteristic function of the unit ball in $\hat{R}^m$, and let $K \in C^\infty(\hat{R}^m)$ have support in $|x| \leqslant \frac{1}{2}$, moreover $K(x) \geqslant 0$ and $\int K\,dx = 1$. Let $F(x) = F_N(x)$ denote the convolution of the functions

$G(x/(N+\frac{1}{2}))$ and K. Then F belongs to $A(\hat{R}^m)$ and clearly satisfies (i), (ii) and (iii). Defining

$$g(t) = (2\pi)^{-m} \int G(x)e^{it\cdot x}\, dx$$

and k, f likewise as the "backward" Fourier transforms of K, F we see that

$$f(t) = (2\pi)^m(N+\tfrac{1}{2})^m\, g((N+\tfrac{1}{2})t)\, k(t). \tag{1}$$

To complete the proof we must show that

$$\int |f(t)|\, dt = O(N^{-(m-1)/2}). \tag{2}$$

First, we have the well-known estimate

$$g(t) = O(|t|^{-(m+1)/2}), \qquad |t| \to \infty. \tag{3}$$

Indeed, cf. [5, p. 171],

$$g(t) = C_m\, |t|^{-m/2}\, J_{m/2}(|t|)$$

and this, plus the classical estimate $J_{m/2}(|t|) = O(|t|^{-1/2})$, as $|t| \to \infty$, for the Bessel function, yields (3).

Now, from (1), since g and k are bounded on R^m,

$$\int_{|t|\leqslant 1/N} |f(t)|\, dt = O(1). \tag{4}$$

Moreover, using (3), we have for constant A depending only on m,

$$\int_{|t|\geqslant 1/N} |f(t)|\, dt \leqslant AN^m \int_{|t|\geqslant 1/N} (N\,|t|)^{-(m+1)/2}\, |k(t)|\, dt$$
$$\leqslant AN^{(m-1)/2} \int_{R^m} |t|^{-(m+1)/2}\, |k(t)|\, dt$$

and since the last integral is finite (recall that $m > 1$), this estimate, together with (4), yields (3), and the theorem is proved.

Remark. With slight modification, the above argument also yields the familiar estimate $\Lambda_N^{(1)} = O(\log N)$. In the case $m = 1$ we could take for k the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]$, in which case f becomes the familiar "trapezoid kernel."

The estimate for $\Lambda_N^{(m)}$ yields as corollaries a number of convergence theorems for spherical partial sums. Since such deductions involve only very well known techniques, we give only a single illustration, also contained in the results of Il'in [1].

COROLLARY. *Let $f \in C(T^m)$ have a (suitably high order) modulus of smoothness $\omega(f, \delta)$ satisfying $\omega(f, \delta) = o(\delta^{(m-1)/2})$ as $\delta \to 0$. Then, $S_N(f, t)$ converges to f uniformly as $N \to \infty$. The exponent $(m - 1)/2$ is sharp.*

Proof (Outline). The hypothesis on f implies $\operatorname{dist}(f, P_N) = o(N^{-(m-1)/2})$, where P_N denotes the set of linear combinations of the exponentials $e^{in\cdot t}$, $|n| \leq N$. (The proof of this is similar to that of [4, p. 210, Theorem 9.3.3.1]). Now, the reasoning used in proving [4, Theorem 9.3.4.2] plus the fact that the $C(T^m) \to C(T^m)$ norm of the map $f \to S_N$ is $O(N^{-(m-1)/2})$, proves the first assertion in the corollary. The sharpness is a consequence of Il'in's results.

CONCLUDING REMARKS

(1) In connection with the corollary, observe that by the (generalized) Bernstein theorem, $\omega(\delta) = O(\delta^{(m/2)+\epsilon})$ implies the convergence of $\sum |\hat{f}(n)|$. Thus, to ensure *uniform* convergence requires nearly as strong a "Lipschitz" condition as that which ensures *absolute* convergence. This is of course a peculiar feature of the spherical sums, of the fact that $\Lambda_N^{(m)}$ is "so large."

(2) There remain several interesting problems about the $\Lambda_N^{(m)}$, especially

(i) Does $\Lambda_N^{(m)}$ increase monotonically with N (as is the case when $m = 1$)?

(ii) What is the asymptotic behavior of $\Lambda_N^{(m)}$, as $N \to \infty$, for fixed m?

ACKNOWLEDGMENTS

I wish to thank B. O. Björnestål for several helpful discussions of the matters treated herein. He also called my attention to the works of Mitchell. I wish also to thank Per Sjölin for making me aware of the paper of Il'in.

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